References

[1]Rusnak, I., and Meir, L., "Modern Guidance Law for High Order Autopilot," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 5, 1991, pp. 1056–1058.

[2]Rusnak, I., "Optimal Guidance for High Order Randomly Maneuvering Target and High Order Missile," IEEE International Conference on Control and Applications-89, Jerusalem, Israel, April 1989.

[3]Bryson, A. E., Jr., and Ho, Y. C., *Applied Optimal Control*, Blaisdell, Waltham, MA, 1969, pp. 414–416.

[4]Speyer, J. L., Deyst, J. J., and Jacobson, D. H., "Optimization of Stochastic Linear Systems with Additive Measurement and Process Noise Using Exponential Performance Criteria," *IEEE Transactions on Automatic Control*, Vol. AC-19, Aug. 1974, pp. 358–366.

[5]Speyer, J. L., "An Adaptive Terminal Guidance Scheme Based on an Exponential Cost Criterion with Application to Homing Missile Guidance," *IEEE Transactions on Automatic Control*, Vol. AC-21, June 1976, pp. 371–375.

[6]Rusnak, I., "Guidance Law Based on an Exponential Cost Criterion for High Order Missile and Maneuvering Target," American Control Conf., ACC—1992, Chicago, IL, June 1992 (Paper 91-1096).

[7]Rusnak, I., and Meir, L., "Optimal Guidance for Acceleration Constrained Missile and Maneuvering Target," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-26, No. 4, 1990, pp. 618–624.

[8]Rusnak, I., and Meir, L., "Optimal Guidance for High Order and Acceleration Constrained Missile," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 3, 1991, pp. 589–596.

[9]Rusnak, I., "Guidance Law Based on an Exponential Cost Criterion for High Order Acceleration Constrained Missile and Maneuvering Target," AIAA Paper 93-3782, Aug. 1993.

[10]Fitzgerald, R. J., and Zarchan, P., "Shaping Filters for Randomly Initiated Targets Maneuvers," AIAA Paper 78-1304, 1978.

[11]Fitzgerald, R. J., "Simple Tracking Filters: Closed Form Solutions," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-17, No. 6, 1981, pp. 781–785.

Analytical Solution of Generalized Three-Dimensional Proportional Navigation

Ciann-Dong Yang* and Chi-Ching Yang†

National Cheng Kung University,
Tainan 701, Taiwan, Republic of China

Introduction

BECAUSE of the difficulties in analysis, most analytical studies in missile guidance laws used two-dimensional models, although actual pursuit–evasion motion occurs in a three-dimensional environment. As for three-dimensional studies, earlier work was done by Adler.[1] The main view point of his study was that the three-dimensional relative motion can be described by the principle plane defined by the instantaneous line of sight (LOS) and by the missile's velocity. Shinar et al.[2] used a linearized three-dimensional model to analyze guidance laws. Guelman and Shinar[3] extended the optimal plane guidance law to three-dimensional models and obtained a preliminary version of optimal three-dimensional guidance law. Cochran et al.[4] considered quasi-three-dimensional relative motion and obtained the analytical solutions for a guidance problem.

In this Note, the concept of generalized three-dimensional proportional navigation is introduced and its analytical solution is derived. The three second-order nonlinear differential equations describing the three-dimensional pursuit–evasion scenario are obtained in terms of the unit angular momentum, which is a useful index of measuring the departure tendence of the three-dimensional relative motion from a fixed plane. The existing difficulties in solving this set of coupled nonlinear equations are removed, and the solution shows that many past studies in the two-dimensional proportional navigation are merely special cases of the present framework.

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*Associate Professor, Institute of Aeronautics and Astronautics. Member AIAA.

†Graduate Student, Institute of Aeronautics and Astronautics.

Equations of Three-Dimensional Relative Motion

Consider the spherical coordinates (r, θ, ϕ) with origin fixed at the missile, where r is the relative distance between the missile and the target, and θ and ϕ are azimuths. Let $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ be unit vectors along the coordinate axes (see Fig. 1). Through the principles of kinematics, the three relative acceleration components (a_r, a_θ, a_ϕ) can be expressed by the following set of second-order nonlinear differential equations:

$$\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\cos^2\phi = a_{T_r} - a_{M_r} \equiv a_r \tag{1a}$$

$$r\ddot{\theta}\cos\phi + 2\dot{r}\dot{\theta}\cos\phi - 2r\dot{\phi}\dot{\theta}\sin\phi = a_{T_\theta} - a_{M_\theta} \equiv a_\theta \tag{1b}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} + r\dot{\theta}^2\cos\phi\sin\phi = a_{T_\phi} - a_{M_\phi} \equiv a_\phi \tag{1c}$$

where a_{T_r}, a_{T_θ}, and a_{T_ϕ} are the acceleration components of the target; a_{M_r}, a_{M_θ}, and a_{M_ϕ} are the acceleration components of the missile. To analyze these coupled nonlinear equations, we find that the unit angular momentum of the relative motion is very helpful. The unit angular momentum $\boldsymbol{h}$ for the missile–target relative motion is defined as

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{2}$$

where $\boldsymbol{r} = r\boldsymbol{e}_r$ is the relative displacement along LOS, and

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\cos\phi\boldsymbol{e}_\theta + r\dot{\phi}\boldsymbol{e}_\phi \tag{3}$$

is the relative velocity. Accordingly, if the relative motion occurs within a fixed plane, i.e., $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are in the same plane during the interception, then the direction of $\boldsymbol{h}$ will be constant, being always perpendicular to the plane spanned by $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$. Hence, the variation of the direction of $\boldsymbol{h}$ is a natural measure for the departure tendency of the relative motion from a fixed plane.

Substituting Eq. (3) into Eq. (2), yields the expression for $\boldsymbol{h}$,

$$\boldsymbol{h} = h\boldsymbol{e}_h = r^2(-\dot{\phi}\boldsymbol{e}_\theta + \dot{\theta}\cos\phi\boldsymbol{e}_\phi) \tag{4a}$$

where

$$h = r^2\sqrt{\dot{\phi}^2 + \dot{\theta}^2\cos^2\phi} \tag{4b}$$

is the magnitude of $\boldsymbol{h}$, and

$$\boldsymbol{e}_h = (r^2/h)(-\dot{\phi}\boldsymbol{e}_\theta + \dot{\theta}\cos\phi\boldsymbol{e}_\phi) \tag{4c}$$

is the unit vector along the direction of $\boldsymbol{h}$. According to the definition of $\boldsymbol{e}_h$, we know that $\boldsymbol{e}_h$ is perpendicular to $\boldsymbol{e}_r$. Therefore, we can find another unit vector $\boldsymbol{e}_h^\perp$ that is perpendicular to both $\boldsymbol{e}_r$ and $\boldsymbol{e}_h$. It is straightforward to verify that

$$\boldsymbol{e}_h^\perp = (r^2/h)(\dot{\theta}\cos\phi\boldsymbol{e}_\theta + \dot{\phi}\boldsymbol{e}_\phi) \tag{5}$$

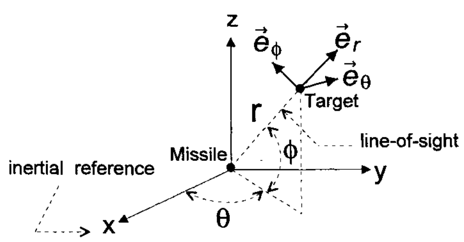

Fig. 1 Three-dimensional pursuit geometry.

The set of unit vectors $(e_r, e_h^\perp, e_h)$ constitutes a new moving coordinate system, which is called h coordinates here. In terms of the h coordinates, Eqs. (1) can be rewritten as

$$\ddot{r} - (h^2/r^3) = a_r \tag{6a}$$

$$\dot{h} = a_h^\perp \tag{6b}$$

$$\dot{e}_h = -(a_h/h)e_h \tag{6c}$$

where the relative acceleration $\boldsymbol{a}$ has been expanded in the h coordinates: $a_r e_r + a_h^\perp e_h^\perp + a_h e_h$ with

$$a_h^\perp = (r^3/h)(a_\phi \dot{\phi} + a_\theta \dot{\theta}\cos\phi)$$
$$a_h = (r^3/h)(a_\phi \dot{\theta}\cos\phi - a_\theta \dot{\phi}) \tag{7}$$

Equations (6) are equivalent to Eqs. (1), but Eqs. (6) have the advantage of decoupling the radial motion from the tangential motion as can be seen in the following discussion.

Generalized Three-Dimensional Proportional Navigation

In the two-dimensional situation, proportional navigation is so called because acceleration of the missile is proportional to the angular velocity $\boldsymbol{\Omega}$ of LOS ($\boldsymbol{r}$) and proportional to the closing velocity $\dot{r}e_r$. In the three-dimensional case, the commanded missile acceleration $\boldsymbol{a}_M$ can be readily extended to the vector form

$$\boldsymbol{a}_M = \lambda \dot{r} e_r \times \boldsymbol{\Omega} \tag{8a}$$

where the angular velocity $\boldsymbol{\Omega}$ of $\boldsymbol{r}$ can be found from the relation

$$\boldsymbol{\Omega} = (\boldsymbol{r} \times \dot{\boldsymbol{r}})/r^2 = \boldsymbol{h}/r^2$$
$$= -\dot{\phi} e_\theta + \dot{\theta}\cos\phi e_\phi \tag{8b}$$

The difference between the angular velocity $\boldsymbol{\Omega}$ of the LOS and the angular velocity $\boldsymbol{\omega}$ of the moving spherical coordinate should be noted. The coordinate $\boldsymbol{\omega}$ can be expressed as

$$\boldsymbol{\omega} = \dot{\theta}\sin\phi e_r - \dot{\phi} e_\theta + \dot{\theta}\cos\phi e_\phi \tag{8c}$$

From Eqs. (8b) and (8c) we see that $\boldsymbol{\Omega}$ is only a component of $\boldsymbol{\omega}$. The reason is that LOS angular velocity $\boldsymbol{\Omega}$ is normal to the instantaneous plane where the relative motion occurs; hence, $\boldsymbol{\Omega}$ does not include the term $\dot{\theta}\sin\phi e_r$, which describes the self-spin motion of the instantaneous plane. Therefore, using $\boldsymbol{\Omega}$, instead of $\boldsymbol{\omega}$, in Eq. (8a) to generate commanded missile acceleration may lose some information about the spin motion of the instantaneous plane. On the other hand, the use of the closing speed $\dot{r}$ in the guidance law (8a) concentrates on the evasion of the target along the LOS direction. Since $\dot{r}$ is merely the e_r component of the relative velocity $\dot{\boldsymbol{r}}$ in Eq. (3), the guidance law (8a) does not include the effect resulting from the relative velocity along the e_ϕ and e_θ directions.

In conjunction with these observations, a general three-dimensional version of proportional navigation is proposed:

$$\boldsymbol{a}_M = \lambda \dot{\boldsymbol{r}} \times \boldsymbol{\omega} \tag{9a}$$

Using $\dot{\boldsymbol{r}}$ from Eq. (3) and $\boldsymbol{\omega}$ from Eq. (8c), Eq. (9a) can be expanded to the form

$$\boldsymbol{a}_M = \lambda\left(\frac{h^2}{r^3}e_r - \frac{\dot{r}h}{r^2}e_h^\perp - \frac{h\dot{\theta}\sin\phi}{r}e_h\right) \tag{9b}$$

Many three-dimensional versions of the existing two-dimensional guidance laws are special cases of the guidance law. Three guidance laws are as follows:

$$\boldsymbol{a}_M = \lambda\left(\dot{r}_0 h/r^2\right)e_h^\perp$$

This guidance law corresponds to the three-dimensional version of the two-dimensional true proportional navigation considered by Guelman.[5] The second is

$$\boldsymbol{a}_M = \lambda(\dot{r}h/r^2)e_h^\perp$$

In the two-dimensional case, this guidance law was referred to as realistic true proportional navigation,[4,6] which can be regarded as the true proportional navigation with varying closing speed. The third is

$$\boldsymbol{a}_M = \lambda\left[(h^2/r^3)e_r - (\dot{r}h/r^2)e_h^\perp\right]$$

This guidance law is the three-dimensional version of the two-dimensional ideal proportional navigation proposed by Yuan and Chern.[7]

These three guidance laws in their vector forms, though, can be equally applied to three-dimensional interception, the resulting interceptor–target relative motions indeed do not occur in three-dimensional space, but are confined in a fixed plane. These types of guidance laws are called quasi-three-dimensional guidance laws. For the generalized three-dimensional proportional navigation defined in Eqs. (9), e_h is a time-varying vector, whereas for quasi-three-dimensional guidance laws, e_h is a constant vector. In the following we will assume a nonmaneuvering target to derive the analytical solution for a missile guided by the commanded acceleration (9b).

The first step in the solution process is to substitute Eq. (8b) into Eqs. (6) to yield the governing equations

$$\ddot{r} - (h^2/r^3) = -\lambda(h^2/r^3) \tag{10a}$$

$$\dot{h} = \lambda(\dot{r}h/r) \tag{10b}$$

$$\dot{e}_h = -\lambda\dot{\theta}\sin\phi e_h^\perp \tag{10c}$$

where we can see that the variables r and h are decoupled with the azimuthes ϕ and θ so that r and h can be solved from Eqs. (10a) and (10b) without considering Eq. (10c). On the contrary, if we substitute Eq. (9b) into Eqs. (1), as is the common procedure used in the literature, the three variables r, ϕ, and θ will be coupled, making it difficult to have a closed-form solution. This decoupling effect is the main advantage of using Eqs. (6), instead of Eqs. (1), to describe the relative motions.

Equations (10a) and (10b) can be solved together, leading to the following results:

$$h = h_0(r/r_0)^\lambda \tag{11a}$$

$$\dot{r} = -\sqrt{\dot{r}_0^2 + \left(h_0^2/r_0^2\right) - \left(h_0^2/r_0^{2\lambda}\right)r^{2(\lambda-1)}} \tag{11b}$$

Since the commanded acceleration (9a) is normal to $\dot{\boldsymbol{r}}$, we can show that the relative speed $V(t) = |\dot{\boldsymbol{r}}|$ remains constant during the interception. This can be checked by substituting Eqs. (11) into Eq. (3) to yield $V(t) = \sqrt{(\dot{r}_0^2 + h_0^2/r_0^2)} = V_0 = \text{const.}$

Equations (1) consist of three second-order differential equations; hence, six initial conditions shall be assigned to determine the solution uniquely. These six initial conditions are chosen as $r_0, \dot{r}_0, \theta_0, \dot{\theta}_0, \phi_0$, and $\dot{\phi}_0$. Without loss of generality, we can choose the inertial reference line as the initial LOS such that $\theta_0 = \phi_0 = 0$. To have the analytical solution independent of the physical units, a nondimensionalized process is required before we proceed further. Dimensionless variables are defined as $\rho = r/r_0$, $\tau = t/(r_0/V_0)$, and $\bar{h} = h/(r_0 V_0)$. Using these dimensionless variables in Eqs. (11), we have

$$\bar{h} = \bar{h}_0 \rho^\lambda \tag{12a}$$

$$\tau = -\int_1^\rho \frac{\mathrm{d}\rho}{\sqrt{1 - \bar{h}_0^2 \rho^{2\lambda-2}}} \tag{12b}$$

The second step in the solution process is to find the azimuths $\theta(t)$ and $\phi(t)$. This step starts from Eq. (6c). Substituting Eq. (4c) into Eq. (6c) and using the transformation between the two coordinate systems (e_r, e_θ, e_ϕ) and $(e_r, e_h, e_h^\perp)$, we obtain the key equation characterizing the behavior of $\theta(t)$ and $\phi(t)$ as

$$\frac{\mathrm{d}(\dot{\theta}\cos\phi)}{-\dot{\theta}\cos\phi} + \frac{\mathrm{d}(\dot{\phi}^2 + \dot{\theta}^2\cos^2\phi)}{2(\dot{\phi}^2 + \dot{\theta}^2\cos^2\phi)} + (1-\lambda)\tan\phi\,\mathrm{d}\phi = 0$$

The integration gives

$$\frac{\dot\phi^2+\dot\theta^2\cos^2\phi}{\dot\theta^2\cos^{4-2\lambda}\phi}=\frac{\dot\phi_0^2+\dot\theta_0^2\cos^2\phi_0}{\dot\theta_0^2\cos^{4-2\lambda}\phi_0}=l^2 \tag{13}$$

where l is an integration constant. This equation can be solved for $\dot\phi^2$ in terms of $\dot\theta^2$ as

$$\left(\frac{d\phi}{d\tau}\right)^2=(l^2\cos^{4-2\lambda}\phi-\cos^2\phi)\left(\frac{d\theta}{d\tau}\right)^2 \tag{14}$$

Equations (4b) and (14) can be solved together for $d\phi/d\tau$ and $d\theta/d\tau$,

$$\frac{d\theta}{d\tau}=\operatorname{sign}(\dot\theta_0)(\bar h_0/l)(\rho\cos\phi)^{\lambda-2} \tag{15}$$

$$\frac{d\phi}{d\tau}=\operatorname{sign}(\dot\phi_0)\,\bar h_0\rho^{\lambda-2}\frac{\sqrt{l^2\cos^{2-2\lambda}\phi-1}}{l\cos^{1-\lambda}\phi} \tag{16}$$

and Eq. (16) can be further integrated with the help of Eq. (12b) to establish the relation between ϕ and ρ

$$\int_0^\phi\frac{d\phi}{\sqrt{1-l^{-2}\cos^{2\lambda-2}\phi}}=-\operatorname{sign}(\dot\phi_0)\int_1^\rho\frac{\bar h_0\rho^{\lambda-2}\,d\rho}{\sqrt{1-\bar h_0^2\rho^{2\lambda-2}}} \tag{17}$$

On the other hand, θ is related to ϕ via Eq. (14), which, after integration, gives

$$\theta=\operatorname{sign}\left(\frac{\dot\theta_0}{\dot\phi_0}\right)\int_0^\phi\frac{d\phi}{\cos^{2-\lambda}\phi\sqrt{l^2-\cos^{2\lambda-2}\phi}} \tag{18}$$

The magnitude of the missile's acceleration can be evaluated from Eqs. (9b), (15), and (16):

$$\frac{a_M}{V_0^2/r_0}=\lambda\bar h_0\rho^{\lambda-2}\sqrt{1+\left(\frac{\bar h_0\rho^{\lambda-1}\sin\phi}{l\cos^{2-\lambda}\phi}\right)^2} \tag{19}$$

From Eq. (19) we can see that the navigation constant λ must be greater than 2 to guarantee the finite acceleration requirement as ρ approaches to zero.

In summary, the solution $[r(t),\theta(t),\phi(t)]$ of the three coupled nonlinear equations (1) with commanded missile acceleration given by Eq. (9b) is derived in Eqs. (12b), (17), and (18). Equation (12b) relates relative distance r to time t; Eq. (17) relates azimuth ϕ to relative distance r; Eq. (18) relates azimuth θ to azimuth ϕ.

To express the solutions more explicitly, we find that the introduction of the following auxiliary azimuths ψ and η is very helpful. The first auxiliary azimuth ψ is defined as

$$\cos\psi=\bar h_0\rho^{\lambda-1} \tag{20}$$

where according to the definition $\bar h_0=h_0/(r_0V_0)=h_0/\sqrt{(h_0^2+r_0^2\dot r_0^2)}$ we have $0\le\bar h_0\le 1$. Hence, as ρ is decreasing from 1 to 0 during the interception, the auxiliary azimuth ψ is varying from ψ_0 to $\pi/2$, where $\psi_0=\cos^{-1}\bar h_0$. In terms of ψ_0, Eq. (20) can be rewritten as

$$\frac{\cos\psi}{\cos\psi_0}=\rho^{\lambda-1} \tag{21}$$

The second auxiliary azimuth η is defined as

$$\frac{\cos\eta}{\cos\eta_0}=\cos^{\lambda-1}\phi \tag{22}$$

where $\cos\eta_0=1/l=\dot\theta_0^2/(\dot\theta_0^2+\dot\phi_0^2)\le 1$. Instead of time τ and relative distance ρ, we recognize that ψ and η are more appropriate for use as independent variables to describe the relative motion analytically. Here, h, ρ, τ, ϕ, and θ can all be expressed by ψ and η. From Eqs. (12a) and (21), we have

$$\frac{\bar h}{\bar h_0}=\left(\frac{\cos\psi}{\cos\psi_0}\right)^{\lambda/(\lambda-1)} \tag{23}$$

The LOS angular rate possesses a similar expression,

$$\frac{\Omega}{\Omega_0}=\rho^{\lambda-2}=\left(\frac{\cos\psi}{\cos\psi_0}\right)^{(\lambda-2)/(\lambda-1)} \tag{24}$$

and ϕ is related to ψ by substituting Eq. (21) into Eq. (17), leading to the following identity:

$$\psi-\psi_0=(\lambda-1)\operatorname{sign}(\dot\phi_0)E_g(l^{-2},2\lambda-2,\phi) \tag{25}$$

where the generalized elliptic function E_g is defined as

$$E_g(a,b,c)=\int_0^c\frac{dx}{\sqrt{1-a\cos^b x}} \tag{26}$$

with a, b, and c being positive real numbers. On the other hand, θ is related to η by substituting Eq. (22) into Eq. (18), leading to the following identity:

$$\theta=\frac{\operatorname{sign}(\dot\theta_0/\dot\phi_0)}{l(\lambda-1)}\times\left[E_g\left(l^{2/(\lambda-1)},\frac{2}{\lambda-1},\eta\right)-E_g\left(l^{2/(\lambda-1)},\frac{2}{\lambda-1},\eta_0\right)\right] \tag{27}$$

The relation between time τ and ψ is established by substituting Eq. (21) into Eq. (12b). After some manipulations, we have

$$\tau=\frac{1}{(\lambda-1)\cos\psi_0}\int_{\psi_0}^{\psi}\left(\frac{\cos\psi}{\cos\psi_0}\right)^{(2-\lambda)/(\lambda-1)}d\psi \tag{28}$$

The required time of interception is attained by letting $\psi=\pi/2$ in the upper limit of the integration. When the initial closing rate $\dot r_0$ is zero, we have $\bar h_0=1$, i.e., $\psi_0=0$. In this case, τ_f has a cleaner expression,

$$\tau_f=\frac{\sqrt{\pi}}{2\lambda-2}\frac{\Gamma[1/(2\lambda-2)]}{\Gamma[\lambda/(2\lambda-2)]} \tag{29}$$

where $\Gamma(\cdot)$ is the gamma function.

The analytical solutions in the dimensionless form are determined uniquely by two parameters $\bar h_0$ and l (ψ_0 and η_0, equivalently), which have the following physical interpretation:

$$\bar h_0=\frac{r_0\sqrt{\dot\phi_0^2+\dot\theta_0^2}}{\sqrt{\dot r_0^2+r_0^2(\dot\phi_0^2+\dot\theta_0^2)}}=\frac{\text{tangential relative velocity}}{\text{total relative velocity}}$$

$$l^{-1}=\frac{\dot\theta_0}{\sqrt{\dot\phi_0^2+\dot\theta_0^2}}=\frac{\text{tangential velocity in the }\theta\text{ direction}}{\text{total tangential velocity}}$$

From Eq. (28), we can verify that $d\tau_f/d\bar h_0>0$ (or $d\tau_f/d\psi_0<0$, equivalently). Hence, as $\bar h_0\to 1$, i.e., in the case that the target is escaping initially almost along the tangential direction, longer time and larger commanded acceleration [see Eq. (19)] are required to accomplish the interception. While $\bar h_0\to 0$, i.e., in the case of tail chase, shorter time and smaller commanded acceleration is needed. The numerical results show that, as expected, when $\bar h_0$ is small (tail-chase case), the influence of the navigation constant λ is quite insignificant; whereas when $\bar h_0$ is close to 1, a large navigation constant can remarkably improve the efficiency of interception.

Conclusions

Three second-order nonlinear differential equations that describe the motion of missiles guided by generalized three-dimensional proportional navigation are solved analytically without any linearization. Based on the introduction of unit angular momentum for the relative motion, a new moving coordinate system is proposed, instead of the conventional moving spherical coordinate system, to decouple the equations of motion so that relative distance and azimuths can be solved independently. It is hoped that this approach provides a systematic framework for analyzing three-dimensional guidance laws in an analytical way and bridges the mathematical gap between two-dimensional guidance laws and three-dimensional guidance laws in the nonlinear setup.

References

[1] Adler, F. P., "Missile Guidance by Three-Dimensional Proportional Navigation," *Journal of Applied Physics*, Vol. 27, May 1956, pp. 500–507.

[2] Shinar, J., Rotsztein, Y., and Bezner, E., "Analysis of Three-Dimensional Optimal Evasion with Linearized Kinematics," *Journal of Guidance and Control*, Vol. 2, No. 5, 1979, pp. 353–360.

[3] Guelman, M., and Shinar, J., "Optimal Guidance Law in the Plane," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 4, 1984, pp. 471–476.

[4] Cochran, J. E., No, T. S., and Thaxton, D. G., "Analytical Solutions to a Guidance Problem," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 1, 1991, pp. 117–122.

[5] Guelman, M., "Closed Form Solution of True Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-12, July 1976, pp. 472–482.

[6] Yuan, P. J., and Chern, J. S., "Solutions of True Proportional Navigation for Maneuvering and Nonmaneuvering Targets," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 1, 1992, pp. 268–271.

[7] Yuan, P. J., and Chern, J. S., "Ideal Proportional Navigation," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, 1992, pp. 1161–1165.

Method for Improving Autopilot Lag Compensation in Intercept Guidance

Thomas R. Blackburn*

Hughes Missile Systems Company, Tucson, Arizona 85734-1337

Introduction

PROPORTIONAL guidance continues to be an important concept for intercept guidance. Practical, energy-efficient engagement transient properties are an important attribute. In its simplest form it is expressed as

$$a_c = \lambda V_c \dot{\varepsilon} \tag{1}$$

where a_c is commanded lateral acceleration, λ the guidance gain, V_c the closing velocity, and $\dot{\varepsilon}$ the inertial line-of-sight rate of the target with respect to the interceptor. Problems can arise in the intercept end game when autopilot lag becomes significant. This shortcoming was ameliorated with the intercept guidance quadratic optimal solution that represented the autopilot as a first-order lag.[1] Performance losses can still be exhibited when the autopilot is not well represented by a first-order lag. The development here presents an intercept guidance closed-form solution that mimics the energy-efficient command transient properties of idealized proportional guidance but accommodates linear autopilot models of arbitrary complexity.

Development

The proportional guidance law can also be expressed in the form[2]

$$a_c = \left(\lambda / t_g^2\right) M \tag{2}$$

where M is the zero-effort miss and t_g is time to intercept. This miss consists of

$$M = p_T - p_M \tag{3}$$

where p_T is the target predicted lateral position at the estimated intercept time (t_g). Here p_M is the vehicle response to its present state, commonly referred to as the zero-effort response. Acceleration profiles for proportional guidance assume the form[2]

$$a_c(t) = a_c(0)(1 - t/t_g)^{\lambda - 2}; \qquad 0 \le t \le t_g \tag{4}$$

Let $f(t)$ represent the interceptor position impulse response to commanded acceleration. The particular solution to the command history response must equal the predicted miss to make intercept. This requirement is expressed by the convolution integral

$$\int_0^{t_g} a_c(t_g - \tau) f(\tau)\, d\tau = M \tag{5}$$

Substituting Eq. (4) into Eq. (5) and rearranging,

$$a_c(0) = \frac{t_g^{\lambda-2} M}{\int_0^{t_g} \tau^{\lambda-2} f(\tau)\, d\tau} \tag{6}$$

The variable $a_c(0)$, (a_c) is available on each computing cycle (time zero) to set to satisfy this equation. Integration by parts in Eq. (6) yields

$$a_c = M/F; \qquad \lambda = 2 \tag{7}$$

$$a_c = \frac{t_g M}{t_g F - E}; \qquad \lambda = 3 \tag{8}$$

$$a_c = \frac{t_g^2 M}{2D - 2t_g E + t_g^2 F}; \qquad \lambda = 4 \tag{9}$$

where

$$F \triangleq \int_0^{t_g} f(\tau)\, d\tau \tag{10}$$

$$E \triangleq \int_0^{t_g} F(\tau)\, d\tau \tag{11}$$

$$D \triangleq \int_0^{t_g} E(\tau)\, d\tau \tag{12}$$

Equations (7–9) can be consolidated into

$$a_c = \frac{t_g^{\lambda-2} M}{\left\{(\lambda-2)\left[(\lambda-3)D - t_g^{\lambda-3}E\right] + t_g^{\lambda-2}F\right\}} \tag{13}$$

for a λ of 2, 3, or 4.

Define

$$\lambda^*(\lambda, t_g) \triangleq \frac{t_g^{\lambda}}{\left\{(\lambda-2)\left[(\lambda-3)D - t_g^{\lambda-3}E\right] + t_g^{\lambda-2}F\right\}} \tag{14a}$$

$$\lambda = 2, 3, 4 \tag{14b}$$

Then, substitution into Eq. (13) yields

$$a_c = \left(\lambda^* / t_g^2\right) M \tag{15}$$

and λ^* can be used interchangeably with λ in Eq. (1) or Eq. (2). Expression (14) is the central contribution of this Engineering Note. It will be referred to as profiled guidance in the following discussion.

Bryson[3] has shown that, assuming a vehicle model has no command-response lag, proportional guidance with a gain of three is optimal. The profiled solution assuming a λ of 3 is therefore of the most interest and is developed further.

Example 1

The profiled solution will now be applied to the system with the first-order-lag autopilot defined by the Laplace form

$$f(s) = \frac{1}{s^2(1 + Ts)} \tag{16}$$

and compared with the optimal[1] solution.

The impulse response of this system is

$$f(t) = Te^{-t/T} + t - T \tag{17}$$

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*Senior Engineer Scientist, Guidance and Control Department, P.O. Box 11337. Senior Member AIAA.